CHAPTER

5 Magnetostatics

5.1 ■ THE LORENTZ FORCE LAW

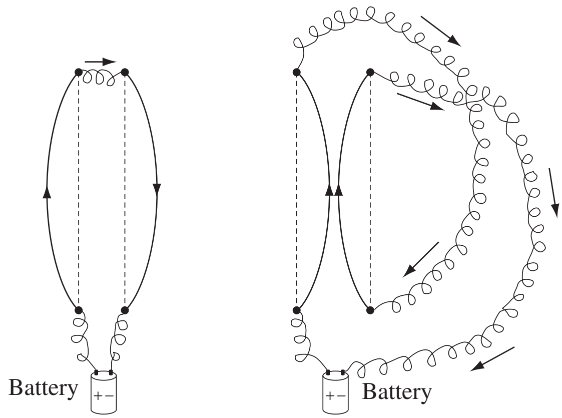
5.1.1 ■ Magnetic Fields

Remember the basic problem of classical electrodynamics: We have a collection of charges $q_1, q_2, q_3, \ldots$ (the "source" charges), and we want to calculate the force they exert on some other charge Q (the "test" charge). (See Fig. 5.1.) According to the principle of superposition, it is sufficient to find the force of a *single* source charge—the total is then the vector sum of all the individual forces. Up to now, we have confined our attention to the simplest case, *electrostatics*, in which the source charge is *at rest* (though the test charge need not be). The time has come to consider the forces between charges *in motion*.

To give you some sense of what is in store, imagine that I set up the following demonstration: Two wires hang from the ceiling, a few centimeters apart; when I turn on a current, so that it passes up one wire and back down the other, the wires jump apart—they evidently repel one another (Fig. 5.2(a)). How do we explain this? You might suppose that the battery (or whatever drives the current) is actually charging up the wire, and that the force is simply due to the electrical repulsion of like charges. But this is incorrect. I could hold up a test charge near these wires, and there would be *no* force on it,[1] for the wires are in fact electrically neutral. (It's true that electrons are flowing down the line—that's what a current *is*—but there are just as many stationary plus charges as moving minus charges on any given segment.) Moreover, if I hook up my demonstration so as to make the current flow up *both* wires (Fig. 5.2(b)), they are found to *attract!*

FIGURE 5.1

[1]This is not precisely true, as we shall see in Prob. 7.43.



(a) Currents in opposite directions repel.

(b) Currents in same directions attract.

FIGURE 5.2

Whatever force accounts for the attraction of parallel currents and the repulsion of antiparallel ones is *not* electrostatic in nature. It is our first encounter with a *magnetic* force. Whereas a *stationary* charge produces only an electric field **E** in the space around it, a *moving* charge generates, in addition, a magnetic field **B**. In fact, magnetic fields are a lot easier to detect, in practice—all you need is a Boy Scout compass. How these devices work is irrelevant at the moment; it is enough to know that the needle points in the direction of the local magnetic field. Ordinarily, this means *north*, in response to the earth's magnetic field, but in the laboratory, where typical fields may be hundreds of times stronger than that, the compass indicates the direction of whatever magnetic field is present.

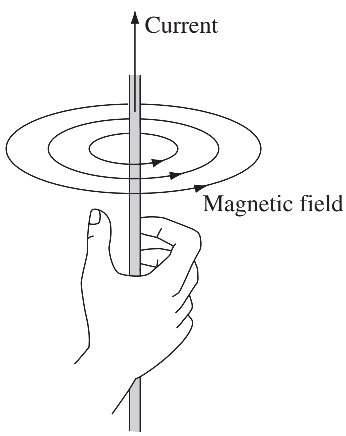


FIGURE 5.3

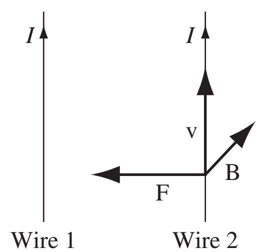


FIGURE 5.4

Now, if you hold up a tiny compass in the vicinity of a current-carrying wire, you quickly discover a very peculiar thing: The field does not point *toward* the wire, nor *away* from it, but rather it *circles around the wire*. In fact, if you grab the wire with your right hand—thumb in the direction of the current—your fingers curl around in the direction of the magnetic field (Fig. 5.3). How can such a field lead to a force of attraction on a nearby parallel current? At the second wire, the magnetic field points *into the page* (Fig. 5.4), the current is *upward*, and yet the resulting force is *to the left*! It's going to take a strange law to account for these directions.

5.1.2 ■ Magnetic Forces

In fact, this combination of directions is just right for a cross product: the magnetic force on a charge Q, moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$, is[2]

$$\boxed{\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B}).} \tag{5.1}$$

This is known as the **Lorentz force law**.[3] In the presence of both electric *and* magnetic fields, the net force on Q would be

$$\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]. \tag{5.2}$$

I do not pretend to have *derived* Eq. 5.1, of course; it is a fundamental axiom of the theory, whose justification is to be found in experiments such as the one I described in Sect. 5.1.1.

Our main job from now on is to calculate the magnetic field $\mathbf{B}$ (and for that matter the electric field $\mathbf{E}$ as well; the rules are more complicated when the source charges are in motion). But before we proceed, it is worthwhile to take a closer look at the Lorentz force law itself; it is a peculiar law, and it leads to some truly bizarre particle trajectories.

Example 5.1. Cyclotron motion. The archtypical motion of a charged particle in a magnetic field is circular, with the magnetic force providing the centripetal acceleration. In Fig. 5.5, a uniform magnetic field points *into* the page; if the charge Q moves counterclockwise, with speed v, around a circle of radius R, the magnetic force points *inward*, and has a fixed magnitude QvB—just right to sustain uniform circular motion:

$$QvB = m\frac{v^2}{R}, \quad \text{or} \quad p = QBR, \tag{5.3}$$

[2]Since $\mathbf{F}$ and $\mathbf{v}$ are vectors, $\mathbf{B}$ is actually a *pseudo*vector.
[3]Actually, it is due to Oliver Heaviside.

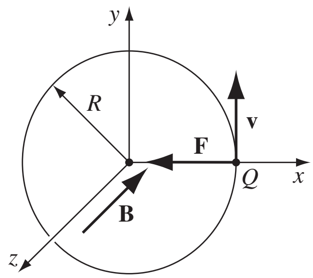


FIGURE 5.5

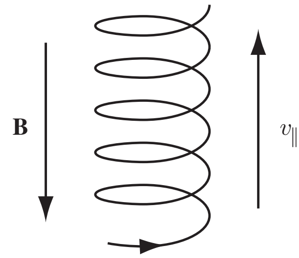


FIGURE 5.6

where m is the particle's mass and $p = mv$ is its momentum. Equation 5.3 is known as the **cyclotron formula** because it describes the motion of a particle in a cyclotron—the first of the modern particle accelerators. It also suggests a simple experimental technique for finding the momentum of a charged particle: send it through a region of known magnetic field, and measure the radius of its trajectory. This is in fact the standard means for determining the momenta of elementary particles.

I assumed that the charge moves in a plane perpendicular to **B**. If it starts out with some additional speed $v_\parallel$ *parallel* to **B**, this component of the motion is unaffected by the magnetic field, and the particle moves in a *helix* (Fig. 5.6). The radius is still given by Eq. 5.3, but the velocity in question is now the component perpendicular to **B**, $v_\perp$.

Example 5.2. Cycloid Motion. A more exotic trajectory occurs if we include a uniform electric field, at right angles to the magnetic one. Suppose, for instance, that **B** points in the x-direction, and **E** in the z-direction, as shown in Fig. 5.7. A positive charge is released from the origin; what path will it follow?

Solution

Let's think it through qualitatively, first. Initially, the particle is at rest, so the magnetic force is zero, and the electric field accelerates the charge in the z-direction. As it picks up speed, a magnetic force develops which, according to Eq. 5.1, pulls the charge around to the right. The faster it goes, the stronger F_{mag} becomes; eventually, it curves the particle back around towards the y axis. At this point the charge is moving *against* the electrical force, so it begins to slow down—the magnetic force then decreases, and the electrical force takes over, bringing the particle to rest at point a, in Fig. 5.7. There the entire process commences anew, carrying the particle over to point b, and so on.

Now let's do it quantitatively. There being no force in the x-direction, the position of the particle at any time t can be described by the vector $(0, y(t), z(t))$; the velocity is therefore

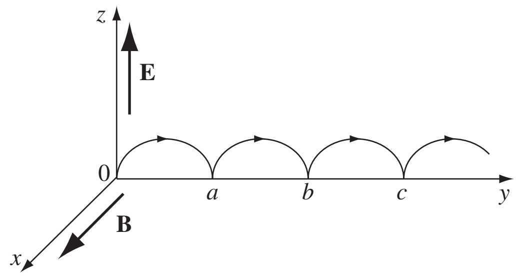


FIGURE 5.7

$$\mathbf{v} = (0, \dot{y}, \dot{z}),$$

where dots indicate time derivatives. Thus

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = B\dot{z}\,\hat{\mathbf{y}} - B\dot{y}\,\hat{\mathbf{z}},$$

and hence, applying Newton's second law,

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = Q(E\,\hat{\mathbf{z}} + B\dot{z}\,\hat{\mathbf{y}} - B\dot{y}\,\hat{\mathbf{z}}) = m\mathbf{a} = m(\ddot{y}\,\hat{\mathbf{y}} + \ddot{z}\,\hat{\mathbf{z}}).$$

Or, treating the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ components separately,

$$QB\dot{z} = m\ddot{y}, \qquad QE - QB\dot{y} = m\ddot{z}.$$

For convenience, let

$$\omega \equiv \frac{QB}{m}. \tag{5.4}$$

(This is the **cyclotron frequency**, at which the particle would revolve in the absence of any electric field.) Then the equations of motion take the form

$$\ddot{y} = \omega\dot{z}, \quad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right). \tag{5.5}$$

Their general solution[4] is

$$\left.\begin{array}{l} y(t) = C_1 \cos\omega t + C_2 \sin\omega t + (E/B)t + C_3, \\ z(t) = C_2 \cos\omega t - C_1 \sin\omega t + C_4. \end{array}\right\} \tag{5.6}$$

[4]As coupled differential equations, they are easily solved by differentiating the first and using the second to eliminate $\ddot{z}$.

But the particle started from rest ($\dot{y}(0) = \dot{z}(0) = 0$), at the origin ($y(0) = z(0) = 0$); these four conditions determine the constants $C_1, C_2, C_3,$ and C_4:

$$y(t) = \frac{E}{\omega B}(\omega t - \sin\omega t), \quad z(t) = \frac{E}{\omega B}(1 - \cos\omega t). \tag{5.7}$$

In this form, the answer is not terribly enlightening, but if we let

$$R \equiv \frac{E}{\omega B}, \tag{5.8}$$

and eliminate the sines and cosines by exploiting the trigonometric identity $\sin^2\omega t + \cos^2\omega t = 1$, we find that

$$(y - R\omega t)^2 + (z - R)^2 = R^2. \tag{5.9}$$

This is the formula for a *circle*, of radius R, whose center $(0, R\omega t, R)$ travels in the y-direction at a constant speed

$$u = \omega R = \frac{E}{B}. \tag{5.10}$$

The particle moves as though it were a spot on the rim of a wheel rolling along the y axis. The curve generated in this way is called a **cycloid**. Notice that the overall motion is *not* in the direction of $\mathbf{E}$, as you might suppose, but perpendicular to it.

One implication of the Lorentz force law (Eq. 5.1) deserves special attention:

Magnetic forces do no work.

For if Q moves an amount $d\mathbf{l} = \mathbf{v}\,dt$, the work done is

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\,dt = 0. \tag{5.11}$$

This follows because $(\mathbf{v} \times \mathbf{B})$ is perpendicular to $\mathbf{v}$, so $(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$. Magnetic forces may alter the *direction* in which a particle moves, but they cannot speed it up or slow it down. The fact that magnetic forces do no work is an elementary and direct consequence of the Lorentz force law, but there are many situations in which it *appears* so manifestly false that one's confidence is bound to waver. When a magnetic crane lifts the carcass of a junked car, for instance, *something* is obviously doing work, and it seems perverse to deny that the magnetic force is responsible. Well, perverse or not, deny it we must, and it can be a very subtle matter to figure out who *does* deserve the credit in such circumstances. We'll see a cute example in the next section, but the full story will have to await Chapter 8.

Problem 5.1 A particle of charge q enters a region of uniform magnetic field $\mathbf{B}$ (pointing *into* the page). The field deflects the particle a distance d above the original line of flight, as shown in Fig. 5.8. Is the charge positive or negative? In terms of a, d, B and q, find the momentum of the particle.

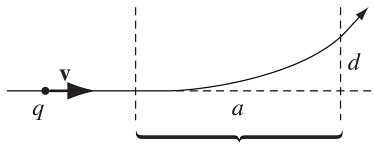


FIGURE 5.8

Problem 5.2 Find and sketch the trajectory of the particle in Ex. 5.2, if it starts at the origin with velocity

(a) $\mathbf{v}(0) = (E/B)\hat{\mathbf{y}}$,

(b) $\mathbf{v}(0) = (E/2B)\hat{\mathbf{y}}$,

(c) $\mathbf{v}(0) = (E/B)(\hat{\mathbf{y}} + \hat{\mathbf{z}})$.

Problem 5.3 In 1897, J. J. Thomson "discovered" the electron by measuring the charge-to-mass ratio of "cathode rays" (actually, streams of electrons, with charge q and mass m) as follows:

(a) First he passed the beam through uniform crossed electric and magnetic fields **E** and **B** (mutually perpendicular, and both of them perpendicular to the beam), and adjusted the electric field until he got zero deflection. What, then, was the speed of the particles (in terms of E and B)?

(b) Then he turned off the electric field, and measured the radius of curvature, R, of the beam, as deflected by the magnetic field alone. In terms of E, B, and R, what is the charge-to-mass ratio (q/m) of the particles?

5.1.3 ■ Currents

The **current** in a wire is the *charge per unit time* passing a given point. By definition, negative charges moving to the left count the same as positive ones to the right. This conveniently reflects the *physical* fact that almost all phenomena involving moving charges depend on the *product* of charge and velocity—if you reverse the signs of q *and* $\mathbf{v}$, you get the same answer, so it doesn't really matter which you have. (The Lorentz force law is a case in point; the Hall effect (Prob. 5.41) is a notorious exception.) In practice, it is ordinarily the negatively charged electrons that do the moving—in the direction *opposite* to the electric current. To avoid the petty complications this entails, I shall often pretend it's the positive charges that move, as in fact everyone assumed they did for a century or so after Benjamin Franklin established his unfortunate convention.[5] Current is measured in coulombs-per-second, or **amperes** (A):

$$1 \text{ A} = 1 \text{ C/s}. \tag{5.12}$$

[5] If we called the electron plus and the proton minus, the problem would never arise. In the context of Franklin's experiments with cat's fur and glass rods, the choice was completely arbitrary.

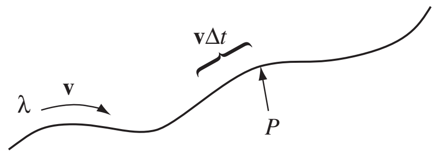


FIGURE 5.9

A line charge λ traveling down a wire at speed v (Fig. 5.9) constitutes a current

$$I = \lambda v, \tag{5.13}$$

because a segment of length $v\Delta t$, carrying charge $\lambda v \Delta t$, passes point P in a time interval Δt. Current is actually a *vector:*

$$\mathbf{I} = \lambda \mathbf{v}. \tag{5.14}$$

Because the path of the flow is dictated by the shape of the wire, one doesn't ordinarily bother to display the direction of $\mathbf{I}$ explicitly,[6] but when it comes to surface and volume currents we cannot afford to be so casual, and for the sake of notational consistency it is a good idea to acknowledge the vectorial character of currents right from the start. A neutral wire, of course, contains as many stationary positive charges as mobile negative ones. The former do not contribute to the current—the charge density λ in Eq. 5.13 refers only to the *moving* charges. In the unusual situation where *both* types move, $\mathbf{I} = \lambda_+ \mathbf{v}_+ + \lambda_- \mathbf{v}_-$.

The magnetic force on a segment of current-carrying wire is

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B}) \lambda\, dl = \int (\mathbf{I} \times \mathbf{B})\, dl. \tag{5.15}$$

Inasmuch as $\mathbf{I}$ and $d\mathbf{l}$ both point in the same direction, we can just as well write this as

$$\boxed{\mathbf{F}_{\text{mag}} = \int I\,(d\mathbf{l} \times \mathbf{B}).} \tag{5.16}$$

Typically, the current is constant (in magnitude) along the wire, and in that case I comes outside the integral:

$$\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B}). \tag{5.17}$$

Example 5.3. A rectangular loop of wire, supporting a mass m, hangs vertically with one end in a uniform magnetic field $\mathbf{B}$, which points into the page in the shaded region of Fig. 5.10. For what current I, in the loop, would the magnetic force upward exactly balance the gravitational force downward?

[6]For the same reason, if you are describing a locomotive constrained to move along a specified track, you would probably speak of its *speed*, rather than its velocity.

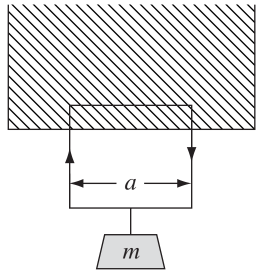


FIGURE 5.10

Solution
First of all, the current must circulate clockwise, in order for ($\mathbf{I} \times \mathbf{B}$) in the horizontal segment to point upward. The force is

$$F_{\text{mag}} = IBa,$$

where a is the width of the loop. (The magnetic forces on the two vertical segments cancel.) For F_{mag} to balance the weight (mg), we must therefore have

$$I = \frac{mg}{Ba}. \tag{5.18}$$

The weight just *hangs* there, suspended in mid-air!

What happens if we now *increase* the current? Then the upward magnetic force *exceeds* the downward force of gravity, and the loop rises, lifting the weight. *Somebody's* doing work, and it sure looks as though the magnetic force is responsible. Indeed, one is tempted to write

$$W_{\text{mag}} = F_{\text{mag}}h = IBah, \tag{5.19}$$

where h is the distance the loop rises. But we know that magnetic forces *never* do work. What's going on here?

Well, when the loop starts to rise, the charges in the wire are no longer moving horizontally—their velocity now acquires an upward component u, the speed of the loop (Fig. 5.11), in addition to the horizontal component w associated with the current ($I = \lambda w$). The magnetic force, which is always perpendicular to the velocity, no longer points straight up, but tilts back. It is perpendicular to the *net* displacement of the charge (which is in the direction of $\mathbf{v}$), and therefore *it does no work on q*. It does have a vertical *component* (qwB); indeed, the net vertical force on all the charge (λa) in the upper segment of the loop is

$$F_{\text{vert}} = \lambda awB = IBa \tag{5.20}$$

(as before); but now it also has a *horizontal* component (quB), which opposes the flow of current. Whoever is in charge of maintaining that current, therefore, must now *push* those charges along, against the backward component of the magnetic force.

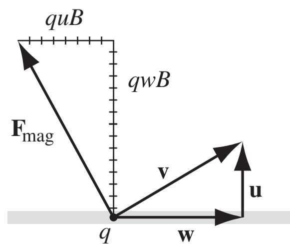


FIGURE 5.11

The total horizontal force on the top segment is

$$F_{\text{horiz}} = \lambda a u B. \tag{5.21}$$

In a time dt, the charges move a (horizontal) distance $w\,dt$, so the work done by this agency (presumably a battery or a generator) is

$$W_{\text{battery}} = \lambda a B \int u w\,dt = IBah,$$

which is precisely what we naïvely attributed to the *magnetic* force in Eq. 5.19. Was work done in this process? Absolutely! Who did it? The battery! What, then, was the role of the magnetic force? Well, it *redirected* the horizontal force of the battery into the *vertical* motion of the loop and the weight.[7]

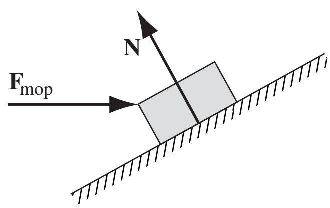


FIGURE 5.12

It may help to consider a mechanical analogy. Imagine you're sliding a trunk up a frictionless ramp, by pushing on it horizontally with a mop (Fig. 5.12). The normal force (**N**) does no work, because it is perpendicular to the displacement. But it *does* have a vertical component (which in fact is what lifts the trunk), and a (backward) horizontal component (which you have to overcome by pushing on the mop). Who is doing the work here? *You* are, obviously—and yet your *force* (which is purely horizontal) is not (at least, not directly) what lifts the box. The

[7] If you like, the *vertical* component of $\mathbf{F}_{\text{mag}}$ does work lifting the car, but the *horizontal* component does equal *negative* work opposing the current. However you look at it, the *net* work done by the magnetic force is *zero*.

normal force plays the same passive (but crucial) role as the magnetic force in Ex. 5.3: while doing no work itself, it *redirects* the efforts of the active agent (you, or the battery, as the case may be), from horizontal to vertical.

When charge flows over a *surface*, we describe it by the **surface current density**, **K**, defined as follows: Consider a "ribbon" of infinitesimal width $dl_\perp$, running parallel to the flow (Fig. 5.13). If the current in this ribbon is $d\mathbf{I}$, the surface current density is

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp}. \tag{5.22}$$

In words, K is the *current per unit width*. In particular, if the (mobile) surface charge density is σ and its velocity is $\mathbf{v}$, then

$$\mathbf{K} = \sigma \mathbf{v}. \tag{5.23}$$

In general, **K** will vary from point to point over the surface, reflecting variations in σ and/or $\mathbf{v}$. The magnetic force on the surface current is

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\sigma \, da = \int (\mathbf{K} \times \mathbf{B}) \, da. \tag{5.24}$$

Caveat: Just as **E** suffers a discontinuity at a surface *charge*, so **B** is discontinuous at a surface *current*. In Eq. 5.24, you must be careful to use the *average* field, just as we did in Sect. 2.5.3.

When the flow of charge is distributed throughout a three-dimensional region, we describe it by the **volume current density**, **J**, defined as follows: Consider a "tube" of infinitesimal cross section $da_\perp$, running parallel to the flow (Fig. 5.14). If the current in this tube is $d\mathbf{I}$, the volume current density is

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp}. \tag{5.25}$$

In words, J is the *current per unit area*. If the (mobile) volume charge density is ρ and the velocity is $\mathbf{v}$, then

$$\mathbf{J} = \rho \mathbf{v}. \tag{5.26}$$

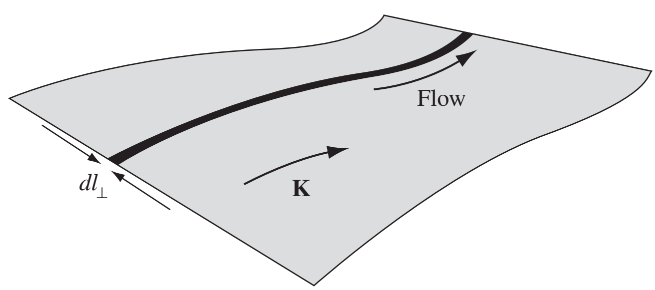


FIGURE 5.13

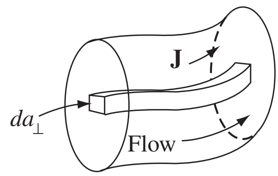


FIGURE 5.14

The magnetic force on a volume current is therefore

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\rho \, d\tau = \int (\mathbf{J} \times \mathbf{B}) \, d\tau. \tag{5.27}$$

Example 5.4.

(a) A current I is uniformly distributed over a wire of circular cross section, with radius a (Fig. 5.15). Find the volume current density J.

Solution
The area (perpendicular to the flow) is πa^2, so

$$J = \frac{I}{\pi a^2}.$$

This was trivial because the current density was uniform.

(b) Suppose the current density in the wire is proportional to the distance from the axis,

$$J = ks$$

(for some constant k). Find the total current in the wire.

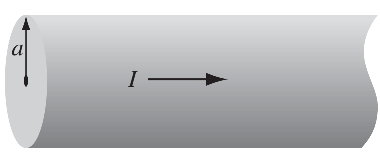


FIGURE 5.15

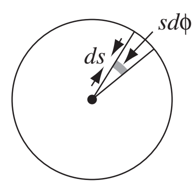


FIGURE 5.16

Solution
Because J varies with s, we must *integrate* Eq. 5.25. The current through the shaded patch (Fig. 5.16) is $J da_\perp$, and $da_\perp = s \, ds \, d\phi$. So

$$I = \int (ks)(s \, ds \, d\phi) = 2\pi k \int_0^a s^2 \, ds = \frac{2\pi k a^3}{3}.$$

According to Eq. 5.25, the total current crossing a surface $\mathcal{S}$ can be written as

$$I = \int_{\mathcal{S}} J\, da_{\perp} = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}. \tag{5.28}$$

(The dot product serves neatly to pick out the appropriate component of $d\mathbf{a}$.) In particular, the charge per unit time leaving a volume $\mathcal{V}$ is

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau.$$

Because charge is conserved, whatever flows out through the surface must come at the expense of what remains inside:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau = -\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t}\right) d\tau.$$

(The minus sign reflects the fact that an *outward* flow *decreases* the charge left in $\mathcal{V}$.) Since this applies to *any* volume, we conclude that

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.} \tag{5.29}$$

This is the precise mathematical statement of local charge conservation; it is called the **continuity equation**.

For future reference, let me summarize the "dictionary" we have implicitly developed for translating equations into the forms appropriate to point, line, surface, and volume currents:

$$\sum_{i=1}^{n} (\quad) q_i \mathbf{v}_i \sim \int_{\text{line}} (\quad) \mathbf{I}\, dl \sim \int_{\text{surface}} (\quad) \mathbf{K}\, da \sim \int_{\text{volume}} (\quad) \mathbf{J}\, d\tau. \tag{5.30}$$

This correspondence, which is analogous to $q \sim \lambda\, dl \sim \sigma\, da \sim \rho\, d\tau$ for the various charge distributions, generates Eqs. 5.15, 5.24, and 5.27 from the original Lorentz force law (5.1).

Problem 5.4 Suppose that the magnetic field in some region has the form

$$\mathbf{B} = kz\,\hat{\mathbf{x}}$$

(where k is a constant). Find the force on a square loop (side a), lying in the yz plane and centered at the origin, if it carries a current I, flowing counterclockwise, when you look down the x axis.

Problem 5.5 A current I flows down a wire of radius a.

(a) If it is uniformly distributed over the surface, what is the surface current density K?

(b) If it is distributed in such a way that the volume current density is inversely proportional to the distance from the axis, what is $J(s)$?

Problem 5.6

(a) A phonograph record carries a uniform density of "static electricity" σ. If it rotates at angular velocity ω, what is the surface current density K at a distance r from the center?

(b) A uniformly charged solid sphere, of radius R and total charge Q, is centered at the origin and spinning at a constant angular velocity ω about the z axis. Find the current density $\mathbf{J}$ at any point (r, θ, ϕ) within the sphere.

Problem 5.7 For a configuration of charges and currents confined within a volume $\mathcal{V}$, show that

$$\int_{\mathcal{V}} \mathbf{J}\, d\tau = d\mathbf{p}/dt, \tag{5.31}$$

where $\mathbf{p}$ is the total dipole moment. [*Hint:* evaluate $\int_{\mathcal{V}} \nabla \cdot (x\mathbf{J})\, d\tau$.]

5

The fields of moving charges

Overview The goal of this chapter is to show that when relativity is combined with our theory of electricity, a necessary conclusion is that a new force, the *magnetic* force, must exist. In nonstatic situations, charge is defined via a surface integral. With this definition, charge is *invariant,* that is, independent of reference frame. Using this invariance, we determine how the electric field transforms between two frames. We then calculate the electric field due to a charge moving with constant velocity; it does *not* equal the spherically symmetric Coulomb field. Interesting field patterns arise in cases where a charge starts or stops.

The main result of this chapter, derived in Section 5.9, is the expression for the force that a moving charge (or a group of moving charges) exerts on another moving charge. On our journey to this result, we will consider setups with increasing complexity. More precisely, in calculating the force on a charge q due to another charge Q, there are four basic cases to consider, depending on the charges' motions. (1) If both charges are stationary in a given frame, then we know from Chapter 1 that Coulomb's law gives the force. (2) If the source Q is moving and q is at rest, then we can use the transformation rule for the electric field mentioned above. (3) If the source Q is at rest and q is moving, then we can use the transformation rule for the force, presented in Appendix G, to show that the Coulomb field gives the force, as you would expect. (4) Finally, the case we are most concerned with: if *both* charges are moving, then we will show in Section 5.9 that a detailed consideration of relativistic effects implies that there exists an additional force that must be added to the electrical force; this is the magnetic force.

In short, the magnetic force is a consequence of Coulomb's law, charge invariance, and relativity.

5.1 From Oersted to Einstein

In the winter of 1819–1820, Hans Christian Oersted was lecturing on electricity, galvanism, and magnetism to advanced students at the University of Copenhagen. *Electricity* meant electrostatics; *galvanism* referred to the effects produced by continuous currents from batteries, a subject opened up by Galvani's chance discovery and the subsequent experiments of Volta; *magnetism* dealt with the already ancient lore of lodestones, compass needles, and the terrestrial magnetic field. It seemed clear to some that there must be a relation between galvanic currents and electric charge, although there was little more direct evidence than the fact that both could cause shocks. On the other hand, magnetism and electricity appeared to have nothing whatever to do with one another. Still, Oersted had a notion, vague perhaps, but tenaciously pursued, that magnetism, like the galvanic current, might be a sort of "hidden form" of electricity. Groping for some manifestation of this, he tried before his class the experiment of passing a galvanic current through a wire that ran above and at right angles to a compass needle (with the compass held horizontal, so that the needle was free to spin in a horizontal plane). It had no effect. After the lecture, something impelled him to try the experiment with a wire running parallel to the compass needle. The needle swung wide – and when the galvanic current was reversed it swung the other way!

The scientific world was more than ready for this revelation. A ferment of experimentation and discovery followed as soon as the word reached other laboratories. Before long, Ampère, Faraday, and others had worked out an essentially complete and exact description of the magnetic action of electric currents. Faraday's crowning discovery of electromagnetic induction came less than 12 years after Oersted's experiment. In the previous two centuries since the publication in 1600 of William Gilbert's great work *De Magnete,* man's understanding of magnetism had advanced not at all. Out of these experimental discoveries there grew the complete classical theory of electromagnetism. Formulated mathematically by Maxwell in the early 1860s, it was triumphantly corroborated by Hertz's demonstration of electromagnetic waves in 1888.

Special relativity has its historical roots in electromagnetism. Lorentz, exploring the electrodynamics of moving charges, was led very close to the final formulation of Einstein. And Einstein's great paper of 1905 was entitled not "Theory of Relativity," but rather "On the Electrodynamics of Moving Bodies." Today we see in the postulates of relativity and their implications a wide framework, one that embraces all physical laws and not solely those of electromagnetism. We expect any complete physical theory to be relativistically invariant. It ought to tell the same story in all inertial frames of reference. As it happened, physics already *had* one relativistically invariant theory – Maxwell's electromagnetic theory – long before the significance of relativistic invariance was recognized. Whether the ideas of special relativity could have evolved in the absence of a complete theory of the electromagnetic field is a question for the historian of science to speculate about; probably it can't be answered. We can only say that the actual history shows rather plainly a path running from Oersted's compass needle to Einstein's postulates.

Still, relativity is not a branch of electromagnetism, nor a consequence of the existence of light. The central postulate of special relativity, which no observation has yet contradicted, is the equivalence of reference frames moving with constant velocity with respect to one another. Indeed, it is possible, without even mentioning light, to derive the formulas of special relativity from nothing more than that postulate and the assumption that all spatial directions are equivalent.[1] The universal constant c then appears in these formulas as a limiting velocity, approached by an energetic particle but never exceeded. Its value can be ascertained by an experiment that does not involve light or anything else that travels at precisely that speed. In other words, we would have special relativity even if electromagnetic waves could not exist.

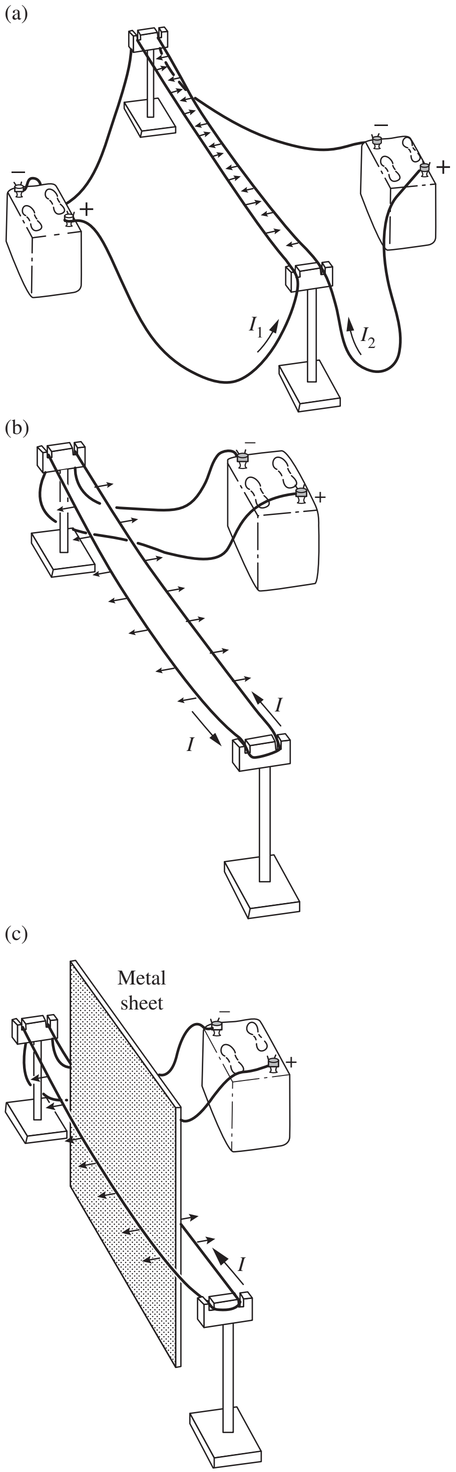
Later in this chapter, we are going to follow the historical path from Oersted to Einstein almost in reverse. We will take special relativity as given, and ask how an electrostatic system of charges and fields looks in another reference frame. In this way we shall find the forces that act on electric charges in motion, including the force that acts between electric currents. Magnetism, seen from this viewpoint, is a relativistic aspect of electricity.[2] But first, let's review some of the phenomena we shall be trying to explain.

5.2 Magnetic forces

Two wires running parallel to one another and carrying currents in the same direction are drawn together. The force on one of the wires, per unit length of wire, is inversely proportional to the distance between the wires (Fig. 5.1(a)). Reversing the direction of one of the currents changes

[1] See Mermin (1984a), in which it is shown that the most general law for the addition of velocities that is consistent with the equivalence of inertial frames must have the form $v = (v_1 + v_2)/(1 + v_1 v_2/c^2)$, identical to our Eq. (G.8) in Appendix G. To discover the value of the constant c in our universe we need only measure with adequate accuracy three lower speeds, v, v_1, and v_2. For references to other articles on the same theme, see also Mermin (1984b).

[2] The earliest exposition of this approach, to our knowledge, is Page (1912). It was natural for Page, writing only seven years after Einstein's revolutionary paper, to consider relativity more in need of confirmation than electrodynamics. His concluding sentence reads: "Viewed from another standpoint, the fact that we have been able, by means of the principle of relativity, to deduce the fundamental relations of electrodynamics from those of electrostatics, may be considered as some confirmation of the principle of relativity."



the force to one of repulsion. Thus the two sections of wire in Fig. 5.1(b), which are part of the same circuit, tend to fly apart. There is some sort of "action at a distance" between the two filaments of steady electric current. It seems to have nothing to do with any static electric charge on the surface of the wire. There may be some such charge and the wires may be at different potentials, but the force we are concerned with depends only on the charge *movement* in the wires, that is, on the two currents. You can put a sheet of metal between the two wires without affecting this force at all (Fig. 5.1(c)). These new forces that come into play when charges are moving are called *magnetic*.

Oersted's compass needle (Fig. 5.2(a)) doesn't look much like a direct-current circuit. We now know, however, as Ampère was the first to suspect, that magnetized iron is full of perpetually moving charges – electric currents on an atomic scale; we will talk about this in detail in Chapter 11. A slender coil of wire with a battery to drive current through it (Fig. 5.2(b)) behaves just like the compass needle under the influence of a nearby current.

Observing the motion of a free charged particle, instead of a wire carrying current, we find the same thing happening. In a cathode ray tube, electrons that would otherwise follow a straight path are deflected toward or away from an external current-carrying wire, depending on the relative direction of the current in that wire (Fig. 5.3). This interaction of currents and other moving charges can be described by introducing a *magnetic field*. (The electric field, remember, was simply a way of describing the action at a distance between stationary charges that is expressed in Coulomb's law.) We say that an electric current has associated with it a magnetic field that pervades the surrounding space. Some other current, or any moving charged particle that finds itself in this field, experiences a force proportional to the strength of the magnetic field in that locality. The force is always perpendicular to the velocity, for a charged particle. The entire force on a particle carrying charge q is given by

$$\boxed{\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}} \tag{5.1}$$

where **B** is the magnetic field.[3]

Figure 5.1.
(a) Parallel wires carrying currents in the same direction are pulled together. (b) Parallel wires carrying currents in opposite directions are pushed apart. (c) These forces are not affected by putting a metal plate between the wires.

[3] Here we make use of the vector product, or *cross product,* of two vectors. A reminder: the vector $\mathbf{v} \times \mathbf{B}$ is a vector perpendicular to both $\mathbf{v}$ and $\mathbf{B}$ and of magnitude $vB\sin\theta$, where θ is the angle between the directions of $\mathbf{v}$ and $\mathbf{B}$. A right-hand rule determines the sense of the direction of the vector $\mathbf{v} \times \mathbf{B}$. In our Cartesian coordinates, $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ and $\mathbf{v} \times \mathbf{B} = \hat{\mathbf{x}}(v_yB_z - v_zB_y) + \hat{\mathbf{y}}(v_zB_x - v_xB_z) + \hat{\mathbf{z}}(v_xB_y - v_yB_x)$.

We shall take Eq. (5.1) as the definition of **B**. All that concerns us now is that the magnetic field strength is a vector that determines the velocity-proportional part of the force on a moving charge. In other words, the command, "Measure the direction and magnitude of the vector **B** at such and such a place," calls for the following operations: Take a particle of known charge q. Measure the force on q at rest, to determine **E**. Then measure the force on the particle when its velocity is **v**; repeat with **v** in some other direction. Now find a **B** that will make Eq. (5.1) fit all these results; that is the magnetic field at the place in question.

Clearly this doesn't *explain* anything. Why does Eq. (5.1) work? Why can we always find a **B** that is consistent with this simple relation, for all possible velocities? We want to understand why there is a velocity-proportional force. It is really most remarkable that this force is strictly proportional to v, and that the effect of the electric field does not depend on v at all! In the following pages we'll see how this comes about. It will turn out that a field **B** with these properties *must* exist if the forces between electric charges obey the postulates of special relativity. Seen from this point of view, magnetic forces are a relativistic aspect of charge in motion.

A review of the essential ideas and formulas of special relativity is provided in Appendix G. This would be a good time to read through it.

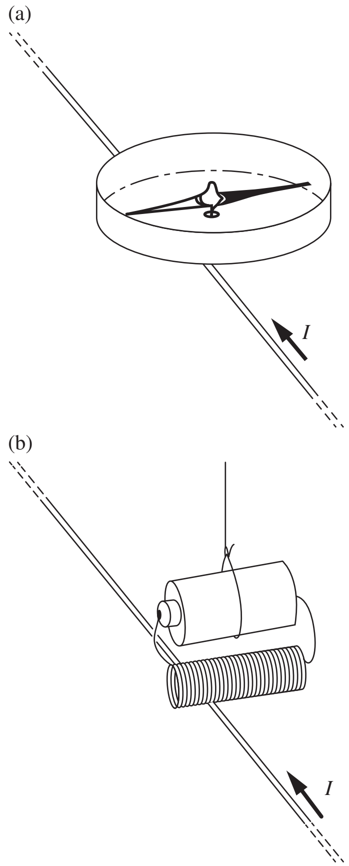


Figure 5.2.
A compass needle (a) and a coil of wire carrying current (b) are similarly influenced by current in a nearby conductor. The direction of the current I is understood to be that in which positive ions would be moving if they were the carriers of the current. In the earth's magnetic field the black end of the compass would point north.

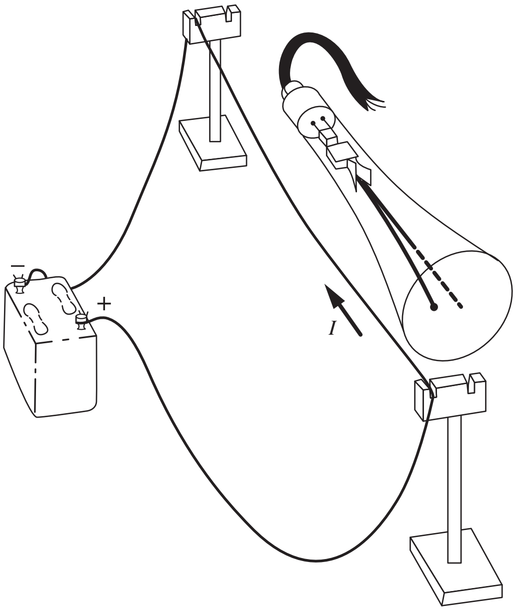


Figure 5.3.
An example of the attraction of currents in the same direction. Compare with Fig. 5.1(a). We can also describe it as the deflection of an electron beam by a magnetic field.